UNIT-IV

Connected Spaces

Defn:

Let X be a topological space . A seperation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X.

Defn:

The space X is said to be connected if there does not exist a separation of X.

Lemma: 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint non-empty sets A & B, whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof:

Let A & B form a separation of Y.

Then A is both open & closed in Y.

Since A is closed in Y, $A=\bar{A} \cap Y$ where $\bar{A}$ is the closure of A in X.

Now, $A \cap B = (\bar{A} \cap Y) \cap B$

$= \bar{A} \cap (Y \cap B)$

$\bar{A} \cap B = \emptyset$

Since $\bar{A}$ contains all the limit points of A, we see that B contains no limit points of A.

Similarly, we can prove that A contains no limit points of B.

For, B is closed in Y, $B = \bar{B} \cap Y$ where $\bar{B}$ is the closure of B in X.

$A \cap B = A \cap (\bar{B} \cap Y)$

$= (A \cap Y) \cap \bar{B}$

$= A \cap \bar{B}$

$\Rightarrow A \cap B = \emptyset$

$\therefore \bar{B}$ contains all the limit points of B.

We see that A contains no limit points of B.

Conversely,

Scanned with CamScanner

Suppose A & B are non-empty disjoint sets whose union is Y, neither of which contains a limit points of the other.

Then $\bar{A} \cap B = \emptyset$ & $A \cap \bar{B} = \emptyset$

Now, $\bar{A} \cap Y = \bar{A} \cap (A \cup B)$

$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$

$= A \cup \emptyset$

$\bar{A} \cap Y = A$ $\quad [\because A \subset \bar{A} \Rightarrow A \cap \bar{A} = A]$

$\therefore$ A is closed in Y.

Similarly, $\bar{B} \cap Y = \bar{B} \cap (A \cup B)$

$= (\bar{B} \cap A) \cup (\bar{B} \cap B)$

$\bar{B} \cap Y = B$ $\quad [\because A \neq \emptyset, B \neq 0, \ Y = A \cup B \text{ and } A \cap B = \emptyset]$

Also A = Y-B & B = Y-A

A & B are open in Y.

A & B form a separation of Y.

Lemma: 23.2

If these sets C and D form a separation of X, and if Y is a connected subspace of X then Y lies entirely within either C or D.

Proof:

Let the sets D and C form a separation of X.

Since C and D are both open in X.

$C \cap Y$ and $D \cap Y$ are both open in Y.

Also, they are disjoint and their union is Y.

Since, $(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y$

$= X \cap Y$

$= Y$

$(C \cap Y) \cap (D \cap Y) = (C \cap D) \cap Y$

$= \emptyset \cap Y$

$= \emptyset$

If both of them non empty, they form a separation of Y.

Since Y is connected, either $C \cap Y = \emptyset$ or $D \cap Y = \emptyset$

Scanned with CamScanner

If $C \cap Y = \emptyset$, then $Y \subset D$

If $D \cap Y = \emptyset$, then $Y \subset C$

Hence $Y \subset C$ or $Y \subset D$

Theorem: 23.3

The union of a collection of connected subspaces of X that have a point in common is connected.

Proof:

Let $A\alpha$ be a collection of connected subspaces of a space X.

Let $Y = \cup A\alpha$

Let they have a point in common.

$\therefore P \in \cap A\alpha$ (say)

$\therefore$ $P \in A\alpha$, for each α

Claim: Y is connected.

Suppose Y is not connected.

Then $Y = C \cup D$, form a separation of Y.

$C \neq \emptyset$, $D \neq \emptyset$, $C \cap D = \emptyset$

Now, $P \in Y = C \cup D$

$\therefore P \in C$ (or) $P \in D$

Assume that $P \in C$

Since each $A\alpha$ is a connected subset of Y, either $A\alpha \subset C$ (or) $A\alpha \subset D$ ($\because$ by lemma 23.2)

Since $P \in C$, $A\alpha$ can not lie in D.

Hence $A\alpha \subset C$, for each α.

$\therefore \cup A\alpha \subset C$

ie) $Y \subset C$

Always $C \subset Y$

$\therefore Y = C$

$\therefore D = \emptyset$

Which is a $\Rightarrow\Leftarrow$ $D \neq \emptyset$

$\therefore Y = \cup A\alpha$ is connected.

Scanned with CamScanner

Theorem: 23.4

Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$ then B is also connected.

Proof:

Let A be a connected subspace of X and Let $A \subset B \subset \bar{A}$

To Prove: B is connected.

Suppose B is not connected.

Then $B = C \cup D$ is a seperation of B.

ie) $(C \cup D = B, C \neq \emptyset, D \neq \emptyset, C \cap D = \emptyset)$

Since A is a connected subspace of B, $A \subset C$ or $A \subset D$ [$\because$ by lemma:23.2]

Suppose $A \subset C$

$\therefore \bar{A} \subset \bar{C}$

$\therefore B \subset \bar{A} \subset \bar{C}$

Since $\bar{C} \cap D = \emptyset$ and $C \cap \bar{D} = \emptyset$ ($\because$ by lemma 23.1)

Since $\bar{C} \cap D = \emptyset$, we have $\bar{A} \cap D = \emptyset$ ($\because \bar{A} \subset \overline{C)}$

$\therefore B \cap D = \emptyset$ ($\because B \subset \bar{A}$)

Hence $B \subset C$

$\therefore D = \emptyset$

Which is a $\Rightarrow\Leftarrow$ to our assumption.

Hence B is connected.

Theorem: 23.5

The image of a connected space under a continuous map is connected. (or) continuous image of a connected space is connected. (or) P.T connectedness is a topological property.

Proof:

Let X, Y be a topological spaces.

Let $f : X \longrightarrow Y$ be continuous map.

Suppose X is connected.

T.P f(X) is connected.

Take Z=f(X) and consider the subspace topology in Z.
consider the restricted map $g:X \longrightarrow Z$ by $g(x)=f(x)$ for every $x \in X$.

Scanned with CamScanner

Then g is a Surjective continuous map. [$\because$ f *is a* restricted by the range]
Suppose Z is not connected.

Then $Z = C \cup D$ is a separation of Z, where C & D are non-empty disjoint open subset in Z.

Now, $Z = f(X)$

$$X = f^{-1}(Z)$$

$$= g^{-1}(Z) = g^{-1}(C \cup D)$$

$$X = g^{-1}(C) \cup g^{-1}(D)$$

Since $g: X \to Z$ is continuous and C, D are open in Z.

Then $g^{-1}(C)$ & $g^{-1}(D)$ are open in X. [$\because$ f is continuous iff inverse image of the open set is open]

$g^{-1}(C) \cap g^{-1}(D) = \emptyset$ [$\because g^{-1}(C) \cap g^{-1}(D) = g^{-1}(C \cap D) = g^{-1}(\emptyset) = \emptyset$]

Since $g^{-1}(C)$ & $g^{-1}(D)$ form a separation of X, X is not connected.
$\Rightarrow\Leftarrow$ to X is connected.

Hence $Z = f(X)$ is connected.
$\therefore$ continuous image of a connected space is connected.

Scanned with CamScanner

Theorem: 23.6

A finite Cartesian product of connected space is connected.

Proof:

First we prove the theorem for the product of two connected space X and Y.

Claim: X×Y is connected.

Choose a base point a×b(X×Y).

Consider the horizontal slice X×b being homeomorphic to X.

∴The horizontal slice X×b is connected.

Consider the vertical slice x×Y being homeomorphic to Y.

∴The vertical slice x×Y is connected.

As a result each T shaped space T_x=(X×b) ∪ (x×Y) is connected.(It is the union of two connected space that have a common point namely X×b)

Now,

X×Y=$¿ x \in X T_x$

∴ $¿x \in XT_x$ is connected since they have a common point a×b.

∴X×Y is connected.

Next we shall prove that $X_1 \times X_2 \times \ldots\ldots\ldots \times X_n$ is connected ,i=1,2,3.....n

Now we prove the result by induction method when n=2,the result is already proved.

Let us assume that the result is true for n-1.

i.e $X_1 \times X_2 \times X_3 \times \ldots\ldots\ldots X_{n-1}$ is connected.

Since $(X_1 \times X_2 \times \ldots\ldots\ldots X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times X_2 \times \ldots\ldots\ldots X_{n-1})$.

∴ $X_{1 \times} X_2 \times \ldots\ldots X_n$ is connected.

Scanned with CamScanner

Section: 26

Compact spaces

Definition

A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition

A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Lemma: 26.1

Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof:

Suppose Y is compact.

Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be an open covering for Y by sets open in X.

Since A_α is open X for each, $A_\alpha \cap Y$ is open in Y.

Then $\{A_\alpha \cap Y\}_{\alpha \in J}$ is open covering for Y by sets open in Y.

Since Y is compact, this open cover has a finite subcover $A_{\alpha 1} \cap Y$, $A_{\alpha 2} \cap Y$,......$A_{\alpha n} \cap Y$. (say)

Then the corresponding elements $A_{\alpha 1}, A_{\alpha 2}, \ldots. A_{\alpha n}$ of $\mathcal{A}$ covers Y.

Conversely,

Assume the condition

Claim: Y is compact.

Let $\{A_\alpha\}_{\alpha \in J}$ be an open covering of Y by sets open in Y.

Scanned with CamScanner

Now, A_α'=$A_\alpha \cap Y$, where A_α is open in X.

Now, $\{A_\alpha\}$ is a open cover for Y by sets open in X.

By our assumption,

This open cover has a finite subcover $A_{\alpha 1}, A_{\alpha 2}, \ldots. A_{\alpha n}$. (say)

Then the corresponding elements $A'_{\alpha 1}, A'_{\alpha 2}, \ldots\ldots A'_{\alpha n}$ is a finite subsets for Y.

Hence Y is compact.

Theorem: 26.2

Every closed subspace of a compact space is compact.

Proof:

Let Y be a closed subspace of a compact space X.

Claim: Y is compact.

Let A be an open covering for Y by sets open in X.

Since Y is closed, X-Y is open in X.

$\therefore \{A \cup$ {X-Y}} is a open cover for X.

Since X is compact this open cover has a finite subcover.

If this finite subcollection contains the set {X-Y} discard {X-Y}, otherwise leave the subcollection alone.

$\{A_{\alpha 1} \cup (X-Y)\} \cup \{A_{\alpha 2} \cup (X-Y)\} \cup \ldots. \cup \{A_{\alpha n}(X-Y)\}$

$¿(A_{\alpha 1}, A_{\alpha 2}, \ldots.. A_{\alpha n}) \cup (X-Y)$

The resulting collection is a finite subcollection of A that covers Y.

$\therefore$Y is compact.

Theorem: 26.3

Scanned with CamScanner

Every compact subspace of a Hausdorff space is closed.

Proof:

Let X be a hausdorff space and let Y be a compact subspace of X.

i.e. To prove : X-Y is open.

Let $x_0 \in X-Y$.

i.e. $x_0 \in X, x_0 \notin Y$

for each $y \in Y, x_0 \neq y$.

Since X is a Hausdorff space, then there exists neighbourhood U_y and V_y of x_0 and y respectively such that $U_y \cap V_y = \varnothing$ .... (1).

Let $\{V_y / y \in Y\}$ be an open covering for Y.

Since Y is compact, there exist a finite subcover $V_{y1}, V_{y2}, \ldots\ldots V_{yn}$ whose union covers Y.

Let $V = V_{y1} \cup V_{y2} \cup \ldots\ldots \cup V_{yn}$.

Let $U_{y1}, U_{y2}, \ldots\ldots U_{yn}$ be the corresponding neighbourhood of x_0.

Let $U = U_{y1} \cap U_{y2} \cap \ldots\ldots \cap U_{yn}$.

Then U is an neighbourhood of x_0.

Also V is an open set inY.

Claim: $U \cap V = \varnothing$

Suppose $U \cap V \neq \varnothing$.

Choose $z \in U \cap V$.

$\therefore z \in U$ and $z \in V$.

Now, $z \in U \Rightarrow z \in U_{yj}$, for all j=1,2,....n

$z \in V \Rightarrow z \in V_{yj}$, for some j.

$\therefore z \in U_{yj} \cap V_{yj}$

$\therefore U_{yj} \cap V_{yj} \neq \varnothing$ which is a contradiction to (1).

$\therefore U \cap V = \varnothing$.

$\therefore U \subset X - Y$.

$\therefore X - Y$ is open.

Hence Y is closed.

Scanned with CamScanner

THEOREM:26.5

The image of a compact space under a continuous map is compact.

Proof:

Let $f: X \to Y$ be continuous.

Let X be compact.

Claim ; f(X) is compact.

Let $\mathcal{A}$ be an open covering of f(X) by sets open in Y.

Since f is continuous, $\{ f^{-1}(A) / A \in \mathcal{A} \}$ is an open cover for X.

Since X is compact, this open cover has a finite subcover $f^{-1}(A_1), f^{-1}(A_2) \ldots f^{-1}(A_n)$ (say) that covers X.

Then the sets $A_1, A_2, \ldots A_n$ covers f(X).

$\therefore$ f(X) is compact.

THEOREM : 26.6

Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof:

Let $f: X \to Y$ be a bijective continuous function.

T.P : f is a homeomorphism.

It is enough to prove that f^{-1} is continuous.

i.e) to prove that if A is closed in X, then $f^{-1}(A) = f(A)$ is closed in Y.

Let A be closed in X, X is compact (given), A is compact in X.

Since f is continuous, f(A) is compact in Y.

Since Y is hausdorff, f(A) is closed in Y.

$\therefore f^{-1}$ is continuous.

Hence f is a homeomorphism.

LEMMA : 26.8

The tube lemma

17:3

Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times y$, where w is a nbd of x_0 in X.

Proof :

First let us cover $x_0 \times y$ by basis element $U \times V$ lying in N.

Since $x_0 \times y$ is homeomorphic to Y and Y is compact.

$\therefore x_0 \times y$ is compact.

We can cover $x_0 \times y$ by finitely many basis element (say) $U_1 \times V_1, U_2 \times V_2, \dots U_n \times V_n$

Let $W = U_1 \cap U_2 \cap \dots \cap U_n$

Then W is a nbd of x_0.

We claim the basis element $U_i \times V_i$, $i = 1,2,\dots n$ which covers $x_0 \times y$ also covers $W \times Y$

Let $x \times y \in W \times Y$

Consider the point $x_0 \times y$ of the slice $x_0 \times y$ having the same y coordinate that of $x \times y$.

Now, $x_0 \times y \in U_i \times V_i$ for some i.

$\therefore y \in V_i$ and $x \in U_j \ \forall j$.

$\therefore x \times y \in U_i \times V_i$ for some i.

$\therefore x \times y \in (U_1 \times V_1) \cup \dots \cup (U_n \times V_n)$

$\therefore W \times Y \subset (U_1 \times V_1) \cup \dots \cup (U_n \times V_n)$

Since each $U_i \times V_i$ lies in N, $(U_1 \times V_1) \cup (U_2 \times V_2) \cup \dots \cup (U_n \times V_n)$ contained in N.

$\therefore W \times Y$ contained in N.

THEOREM ; 26.7

The product of finitely many compact spaces is compact.

Proof :

We shall prove that the product of two compact spaces is compact.

Then the thm follows by induction for any finite product.

STEP : 1

State and prove tube lemma.

STEP : 2

Let X and Y be two compact spaces.

Scanned with CamScanner

CLAIM : $X \to Y$ is compact.

Let $\mathcal{A}$ be an open covering for $X \times Y$.

Choose any $x_0 \in X$ and fix it.

Consider $x_0 \times y$

Since $x_0 \times y$ is compact, there are finitely many members (say) $A_1, A_2, \ldots, A_m$ which cover $x_0 \times y$

Now define $N = A_1 \cup \ldots \cup A_m$

Then N is an open set containing the slice $x_0 \times y$, there exist a neighbourhood W of x_0 such that $W \times Y \subset N$

$\therefore$ For each $x \in X$, we can choose a neighbourhood W_x of x such that $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

Consider the collection $\{ W_x / x \in X \}$ be an open covering of X.

Since X is compact, there exists a finite subcollection $W_1, W_2, \ldots, W_n$ cover X.

$\therefore X = \cup_{i=1}^{n} W_i$

Now $X \times Y = (W_1 \times Y) \cup \ldots \cup (W_n \times Y)$

$\therefore \mathcal{A}$ has finite subcollection that covers $X \times Y$.

Hence $X \times Y$ is compact.

STEP : 3

We prove that finite product of compact spaces is compact.

We have to prove the result by induction on n.

By step 2; The result is true for $n = 2$

Assume that the result is true for n.

We have to prove the result for $n+1$ $(X_1 \times X_2 \times \ldots X_n) \times X_{n+1}$ is compact.

$\therefore$ The result is true for n+1.

Thus the finite product of compact space is compact.

Defn :

A collection $\mathcal{G}$ of subsets of X is said to have the finite intersection property if for every finite subcollection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{G}$, the intersection $C_1 \cap C_2 \cap \ldots \cap C_n$ is non- empty.

THEOREM : 26 . 9

Scanned with CamScanner

Let X be a topological space .Then X is compact iff for every collection $\mathcal{G}$ of closed sets in X having the finite intersection property. The intersection $c \in \mathcal{G}$ of all the elements of $\mathcal{G}$ is non- empty.

Proof :

Given a collection $\mathcal{A}$ of subsets of X .

Let $\mathcal{G} = \{ X - A / A \in \mathcal{A} \}$ be the collection of their complements .

Then $\mathcal{A}$ is a collection of open sets iff $\mathcal{G}$ is a collection of closed sets .

Claim : 1

The collection $\mathcal{A}$ covers X iff the intersection $c, c \in \mathcal{G}$ of all the elements of $\mathcal{G}$ is empty .

For , $\mathcal{A}$ covers X $\Leftrightarrow \cup_{A} \in \mathcal{A} = X$

$$\Leftrightarrow (\cup_{A} \in \mathcal{A} \quad A)^c = X^c$$

$$\Leftrightarrow \cap_{A^c} \in \mathcal{G} \quad A^c = \emptyset$$

$$\Leftrightarrow \cap_{C} \in \mathcal{G} \quad C = \emptyset$$

Claim : 2

The finite subcollection $\{ A_1, A_2, \ldots, A_n \}$ of $\mathcal{A}$ covers X $\Leftrightarrow$ the intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{G}$ is empty.

For, $\{ A_1, A_2, \ldots, A_n \}$ covers X $\Leftrightarrow \cup_{i=1}^{n} A_i = X$

$$\Leftrightarrow (\cup_{i=1}^{n} A_i)^c = X^c \Rightarrow \cap_{i=1}^{n} A_i^c = \emptyset$$

$$\Rightarrow \cap_{i=1}^{n} C_i = \emptyset$$

Suppose X is compact.

Let $\mathcal{G}$ be a collection of closed sets having finite intersection property.

Claim : $\cap_c \in \mathcal{G} \neq \emptyset$

On the contrary

Suppose $\cap_c \in \mathcal{G} \quad C = \emptyset$

The collection $\mathcal{A} = \{ X - C / c \in \mathcal{G} \}$ is an open covering of X.

Since X is compact, there is a finite subcollection $\{ A_1, A_2, \ldots, A_n \}$ of A also covers X.

Then $C_1 \cap C_2 \cap \ldots C_n = \emptyset$

$\therefore \cap_c \in \mathcal{G} \quad C = \emptyset$

Which is a contradiction to $\mathcal{G}$ has the finite intersection property.

Scanned with CamScanner

$\therefore \cap_{c \in \mathcal{G}} C \neq \emptyset$

Conversely,

Suppose $\cap_{c \in \mathcal{G}} C \neq \emptyset$ for every collection $\mathcal{G}$ of closed sets having finite intersection property.

T.P : X is compact.

Let $\mathcal{A}$ be an open covering of X.

$\mathcal{G} = \{ X - A / A \in \mathcal{A} \}$ is the collection of closed sets s.t $\cap_{c \in \mathcal{G}} C = \emptyset$

Then $\mathcal{G}$ cannot have the finite intersection property.

By claim :2, $\{ A_1, A_2, \ldots\ldots, A_n \}$ covers X.

Thus every open covering X contains a finite subcollection that also covers X.

Hence X is compact.